# Introduction to Petrophysics – Physical basis

# Continuum Physics

Models are the simplified reality, where we keep the most important features and neglect the properties which do not or not substantially influence the examined process. In continuum physics the characteristics of the material are described by continuous functions which is inconsistent with the atomic structure. However in the description of many phenomenas (e.g. elasticity, flows) the atomic and molecular descriptions are not necessary. The soultion is that we introduce the phenomenological description method by averaging the atomic effects and take in so-called material characteristics "constants”. These characteristics are usually non constants they are depend on temperature or other quantities. Thus we obtain a simplified - continuum - model of the material which is applied in many areas of rock mechanics and rock physics. The theory describing the mechanical properties of the continuum, the continuum mechanics is a phenomenological science.

Based on the continuity hypothesis density functions assigned to extensive physical quantities (mass, momentum, energy) are considered mathematically as continuous functions of the location coordinates. Thus e.g. mass-density function  defined as follows

,

where  is a small volume around the point ,  is the implied mass. The boundary transition  is interpereted physically i.e. in the equation



corresponding (2.1) the voulme  is „physically infinitesimal”. The boundary transition  at low volume in mathematical sense leads to that the material belongs to  will be qualitatively different. The boundary transition  can be understood as  tends to a volume  which is quite large on the atomic scale but on the macroscopic scale it is small (small enough to be considered as point-like). It can be seen that the continuum mechanical and atomic description methods can be compatible. In continuum mechanics we get the simplified descripiton of large groups of atoms thus we obtain relative simple equations. We deduce general laws hence further equations characterising the specific properties of the material are always necessary. These are the material equations containing so-called „material constants” (e.g. elastic moduli) which reflect the neglected atomic features. In continuum mechanics the functions representing physical quantities are fractionally continuous, i.e. it may be exist surfaces in the media (eg. layer boundaries) along which the respective quantities suffer finite "hop". Boundary condition equations defined along these surfaces must be met.

## **2.1. Deformations and strains**

After the axiom of the kinematics of deformable bodies the general movement (if it is quite small) of sufficiently small volume of the deformable body can be combined by a translation, a rotation and an extension or contraction took in three orthogonal directions. In the framwork of continuum mechanics the displacement is given by the continuous vector . To illustrate the meaning of the axiom of kinematics let us take up the coordinate system in the point  of the deformable continuum and consider point  (originated from the small volume assumed around ) close to . During the movement of the continuum  pass through point  satisfying the vector equation

,

where  is the displacement vector „connencting” points  and . Assume that fracture surface is not extend between points  and . Then the two (adjacent) points can not move independently of one another, there is a „material relationship" between them defined by the continuum material.

This can be expressed mathematically that the displacement in point  is originated to characteristics refer to point , or in other words the displacement is exerted into series around point :





, (2.1.1)

where … means other "higher parts" appearing in the expansion and index  indicating next to the derivatives refers to that the derivates should take in the origin ().

In the axiom of kinematics we talk about the “displacement of sufficiently small volume”. It means that in equations (2.1.1) the "higher parts" containing the powers and products of coordinates  are negligible, i.e. we live with a linear approximation. In addition, we also assume that the first derivatives are small in the sense that their product and powers are negligible. Herewith (2.1.1) takes the following form

 (2.1.2)

In the followings, we will apply the so-called Einstein's convention with which (2.1.2) can be written as



i.e. if an index (or indices) in an expression occurs twice we should sum from 1 to 3. In the followings the index 0 beside the derivatives  is omitted, so

 (2.1.3)

The derivative tensor  can be divided into symmetric and antisymmetric parts as



With this we obtain the following equation for the (2.1.3) displacements

, (2.1.4)

where  is the same for any points of the small volume taken around, i.e.  means homogeneous translation for the movement of these points



It can be proved that the second part in (2.1.4) represents the rotation of the continuum element. Thus, it is obvious that the third part of (2.1.4) provides the deformation displacements

,

where

 (2.1.6)

Introducing the notation

 (2.1.7)

 (2.1.6) can be reformulated as

, (2.1.8)

where the symmetrical second-order tensor  called deformation tensor.

To clarify the components of the deformation tensor take a material line in unit length along the coordinate axis  of the original coordinate system and denote it as vector . During deformation this transforms to vector

 (2.1.9)

due to equation (2.1.8), according to the equation

 .

Hence the relative expansion is

,

as we limit ourselves to small deformations .

Thus deformation  means the expansion of the section in unit length taken along axis  or in other words the relative expansion measured along axis . Elements  have similar meaning. Elements in the main diagonal of the deformation tensor give the relative expansion of the material line sections falling into axes . To investigate the meanings of the elements outside the main diagonal let us take the unit vector  falling into the direction of the coordinate axis  which after deformation transforms into vector

.

By equation (2.1.9) plus forming the scalar product  and neglecting the squares as well as products of deformations one obtains

.

Applying

,

where  and the angle between the two vectors is



results as

,

where  was used to small angles. Ergo deformation  is the half of the angle change which is suffered by the line section taken in the originally perpendicular directions  and .

Take a prism with a volume  and with edges parallel to the coordinate axes in the undeformed continuum! The volume of the prism generated during deformation will be approximately



i.e. the relative volume change is

.

The sum of the elements in the main diagonal of the deformation tensor (a.k.a. the spur of the deformation tensor or the invariant of the first scalar) means the relative volume change. Contrariwise one can expresses it with equation



 (summarize to!), or on the basis of definition (2.1.7) of the deformation tensor

. (2.1.10)

This quantity is unchanged during coordinate transformation.

To characterize the deformations it is used to introduce the spherical tensor

 (2.1.11)

and the deformation deviatoric tensor

, (2.1.12)

where  is a unit tensor i.e.

.

The name is originated in that the second-order tensor surface ordered to the deformation spherical tensor is sphere. With this tensor the pure volume change can be separated from the deformations. The remaining part of deformations  shows the deviation from the pure volume change, i.e. the so-called distortion. It is obvious according to (2.1.12) that. The decomposition of the deformation tensor

 (2.1.13)

means also the pick apart it to the volume change-free "pure distortion" and the pure volume change.

 The dynamic interpretation of the movement of the continuum requires to introduce the force densities. Experiences show that the forces affecting on the continuum can be divided into two types: volume and surface forces. The volume force  - affecting to the continuum contains the (physically) infinitesimal volume element  took at a given point of the space - can be written as

,

where  is the volumetric force density. The integral of the volumetric force density



gives the force affected to finite volume. The volumetric force density can be calculated otherwise by the definition

.

There are forces that are physically directly proportional not to the volume, but the mass. These can be characterized by the mass force density

,

where is the mass contained in the volume . With which



or otherwise

. (2.1.14)

Another class of forces arising in the continuum are the surface forces. The surface force density can be formulated as

, (2.1.15)

where the boundary transition  can be interpreted as  tends to a so small surface  which is negligible (point-like) in macroscopic point of view but it is very large compared to the atomic cross section. Otherwise equation (2.1.15) can be written as

 . (2.1.16)

In (2.1.16) „index”  implies that the surface force at a given space depends on not only the extent of the surface but its direction - characterized by the normal unit vector  - too. Based on (2.1.16) the force affected on the finite surface *A* can be calculated as

. (2.1.17)

Since the unit normal vector  can point to infinite number of directions apparently the knowlege of infinitely many surface force densities is necessary to provide the surface forces. However it can be proved that

 (2.1.18)

This equation shows that if we know in one point the surface force density  affected on three orthogonal coordinate plane then surface force densities (also known as strains) affected on any  directional surface can be calculated by the help of equation (2.1.18).

Introducing the notations









 (2.1.18) can be written as

 (2.1.19)

(where according to our agreement one has to summarize to *j* from 1 to 3). Ergo after (2.1.19) to characterize the surface force density the second-order tensor  is introduced which is the j-th component of the strain vector affected on the surface supplied with a normal pointing to the direction of the i-th coordinate axis. It can be proved that this tensor is symmetric, i.e.



The elements  in the main diagonal of the tensor are normal directional (tensile or compressive) stresses, the outside elements  are tangential (shear or slip) stresses. Similarly at the deformation tensor one can produce the stress tensor as the sum of the deviator and the stress spherical tensors.

 (2.1.20)

where

, (2.1.21)

and  is the sum of the elements in the main diagonal of the stress tensor.

## **2.2. The equation of motion**

At the deduction of the motion equation of deformable continua the starting pont is Newton's II. law which says that the time-derivate of the impulse of the body equals to the sum of the arose forces

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The impulse of the body can be determined by the formula



where



is the volumetric impulse density, while  is the velocity. The resultant force affecting the body is the sum of the volume and surface forces, i.e.



The integral form of the motion equation can be written as



where ,  are the volume as well as surface moving together with the continuum. To the i-th coordinate of the vector equation one can obtain

. (2.2.1)

To transform the latter equation use the identity



and the Gauss-Osztogradszkij thesis



where  and



denotes - as a formal vector - the i-th row of the stress tensor. Here the equation (2.1.19) was also used. Now the motion equation (2.2.1) can be written as



Hence volume  is arbitrary, from the disappearance of the integral one can infer to the disappearance of the integrand



otherwise

 (2.2.2)

This is the local form of the motion equation of the deformable continuum, also known as the balance equation of the impulse.

In continuum theory equations describing the transport of extensive quantities can be commonly reformulated to the format of the continuity equation. If the bulk density of a quantity is  then the convective current density of the given quantity is denoted as .  represents the conductive (connected to macroscopic motions) current density. Then the balance equation of quantity  is



or if it has sources (or sinks)

 (2.2.3)

where  is the source strength which provides the quantity of  produced or absorbed in unit volume per unit time.

Introducing the convective  and conductive  impulse-current density vectors equation (2.2.2) can be written as

.

Ergo the stress tensor (that of onefold) is physically the conductive impulse current density, while volumetric force density  plays the role of the source strength of the impulse. It is well-known that similar balance equation can be formed to (mass) density  of the j-th component of a fluid compound

,

where the convective mass flow density is . The conductive mass flow density provides a way to decribe the diffusive motions, source strength  refers to the chemical reaction which gives the production of the j-th component. To one-component fluid by neglecting the source strength the continuity equation

 (2.2.4)

provides the balance equation of the mass. By transforming the left side of equation (2.2.2) one can obtain

,

where the identity



was used (where  and  are the continuous function of the three spatial coordinates). By taking into consideration the continuity equation, (2.2.2) can be written as

.

We call the partial derivate  otherwise local, while the operator  is the convective derivate and



is the substantial derivative. Hereby the motion equation can be written as

. (2.2.5)

In solid continua the convection can be negligible thus  and the motion equation is

. (2.2.6)

In vector form

, (2.2.7)

corresponds with the motion equation (2.2.5), while

 (2.2.8)

refers to equation (2.2.6), where *Div* is the sign of the tensor divergence and the double underline denotes the tensor.

The continuity equation (2.2.4) and equation (2.2.5) are the continuum mechanical formulation of the mass conservation and the impulse thesis respectively, i.e. express general (valid for any continuum) law of nature. However there are 10 scalar unknowns in these four scalar equations (assuming the mass forces  as knowns). Thus equations derived from natural basic law are significantly underdetermined, so unambigous solution can not be existed.

To clearly describe dynamically the movement of the continuum more six equations are required which can be obtained on the basis of restrictive conditions took to the material quality of continuum and its elastic properties. These equations are the material equations wrote to the six independent elements of the stress tensor.

## **2.3. Material equations**

Elastic properties of material continua are very diverse. A general material equation which comprise all of this variety, does not exist. Instead, one should highlight from all elastic properties of the investigated medium the most relevant ones and neglect the other "disturbing" circumstances. This can be expressed differently i.e. we create a model. The most important simple and complex material models built from the simple ones will be described in the followings especially considering the rock mechanics and seismic/acoustic aspects.

### 2.3.1. Material equation of the perfectly elastic body, stress dependent elastic parameters

Perfectly elastic body means that stresses depend on the deformations dominant at a given space of the continuum in a given time, i.e.

. (2.3.1)

The function  is generally non-linear. However, very often we deal with small stress change related to small deformations. For example, if an elastic wave propagates in a medium existed in a given stress state, the wave-induced deformation and stress perturbation is very small compared to the characteristics of the original, static load of the medium.

In this case, the function  can be approximated by the linear parts of its power series

, (2.3.2)

where the notations





were introduced. The constants  are named elastic constants which characterize the perfectly elastic body near the undeformed state. Since waves mean small deformation, this time the series expansion (2.3.3) provides good approximation. Elastic parameters  in (2.3.3) form a 6x6 matrix. It can be proved by the help of the energy thesis formulated to continua, this matrix is symmetric. This means that in general case, the elastic properties of the anisotropic continuum can be characterized by 21 independent elastic parameters. The properties of material symmetry can significantly reduce the number of elastic constants. Isotropic continuum can be characterized by two elastic constants. In small deformation interval the model of perfectly elastic body transforms to linearly elastic body.

### 2.3.2. Material equation and equation of motion of Hooke-body

The phenomenological description of anisortopy is very important in rock physics and seismic too. However, the simplification is reasonable in seismic practice the most widely used linearly elastic medium model assumes isotropy. The linearly elastic isotropic body is characterized by only two elastic parameters which can be introduced a number of ways. With thermodynamic considerations the two parameters are the so-called first  and second  Lame coefficients with which the material equation of the linearly elastic isotropic body or Hooke-body can be written as

 (2.3.4)

Hence to the main diagonal of the stress tensor one can obtain the equation

, (2.3.5)

where  is the compression modulus.

Introducing the stress spherical tensor

,

on the basis of (2.3.5) its relationship with the deformation spherical tensor is

. (2.3.6)

The stress deviatoric tensor  after (2.3.4) is

. (2.3.7)

The material characteristic parameters  and  are generally depend on temperature too.

In engineer life instead of Lame coefficients the Young's modulus  and Poisson's number  are often used. In case of uniaxial load (e.g. a thin long rod clamped at one end, its other end is pulled) if  is axial, the tensile is

 (2.3.8)

so the Young's modulus  can be determined directly. In the plane perpendicular to the tension the deformations  have opposite sign and proportional to the relative expansion 



where  is the Poisson's number. The relative volume change is

 (2.3.9)

Since in case of  (stretching) the volume can not decrease  so from (2.3.9) . The equality refers to incompressible materials  (e.g. static load in fluid).

To look for the relationship of parameters  and  write the expression of  (2.3.9) to (2.3.4). At uniaxial load

, (2.3.10)

on the other hand because of  (since there is only one stress component exists)

. (2.3.11)

By comparing the equations (2.3.10) and (2.3.11), as well as (2.3.8)

,



or

,

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One can obtain the motion equation of the linearly elastic isotropic body if one substitutes the material equation (2.3.4) to the general motion equation (2.2.6.). By forming the divergence of the stress tensor (2.3.4) in case of homogeneous medium ( and are independent from location)

,

where the definition (2.1.7) of the deformation tensor was used and one must sum for the same indexes. Since



and

,

the motion equation can be written as

, (2.3.12)

in vectorial form by using (2.1.10)

. (2.3.13)

This equation alias the Lame equation is the motion equation of the linearly elastic homogeneous body (Hooke-body). Mathematically (2.3.13) is an inhomogeneous, second-order nonlinear coupled partial differential equation system. To obtain its unambiguous solution initial and boundary conditions are necessary. Setting the initial value problem means that we require the displacement  and velocity  at each point of the tested  volume in . Boundary conditions require the displacement  and the value of the directional (normal) derivative  at any  time in  points of the surface  bounding volume .

In case of inhomogeneous linearly elastic isotropic body the „Lame coefficients” depend on space:  So the divergence of the stress tensor (2.3.4) can be written in the form

.

With which the motion equation is



or in vectorial form



where  denotes vectorial multiplication.

### 2.3.3. Fluid mechanical material models and their motion equations

Fenomenoligical definition of fluids is based on the experience that the smaller the tangential (shear) stresses occurring in fluids are the slower the deformation is. By extrapolating this observation we consider that continuum as fluid, in which shear stresses do not occur in repose state, i.e. elements outside the main diagonal of the stress tensor are disappeared in every coordinate system. In isotropic fluids, elements in the main diagonal are equal, i.e. the stress tensor in repose state is

,

where  is the scalar pressure.

**Material and motion equation of ideal fluid (Pascal’s body)**

We call that fluid ideal in which shear stresses during motion do not occur, i.e. the stress tensor of the ideal fluid for any deformation is

. (2.3.14)

Since then

,

the stress tensor of the ideal fluid is a spherical tensor. This is another formulation of the well known - from fluid mechanics - Pascal's law, therefore the ideal fluid called otherwise Pascal's body.

Equation (2.3.14) only makes constraint to the format of the stress tensor, but it is not a material equation. The material equation usually connects the stresses with kinematic characteristics. In contrary in fluid mechanics the pressure is investigated in density and temperature dependence. For example if pressure depends on only density

,

we talk albout barotripoic fluid.

Equation (2.3.14) is valid in case of gases too. The state equation of ideal gases can be written as

,

where  is the gas constant and  is the absoulte temperature. The state equation is simpler in case of special change of state. E.g. at isothermal processes

,

while in case of adiabatic change of state

,

where   is the specific heat mesured at constant pressure as well as  is at constant volume, respectively. Based on (2.2.5) and (2.3.14) the motion equation of the ideal fluid is



or in vectorial form

 . (2.3.15)

This equation is the so-called Euler-equation.

**Material and motion equation of the Newtonian fluid**

The ideal fluid model not enables to describe a number of practical problems. It is a general experience that waves absorb in fluids or friction losses occur in fluids during flowing. To explain these phenomena an improved fluid model is required.

At the phenomenological definition of fluids we highlighted that shear stresses are the smaller the slower the deformation is. This means that stresses in fluid originated from friction depend on the swiftness of the deformations, the deformation velocity tensor

 

i.e.

.

In terms of geophysical applications only the isotropic fluids have significance which show linear depencence in deformation velocities. Then (because of the isotropy) writing the tensor  instead of deformations  in the (2.3.4) formula of the tensor , one obtains the material equation

, (2.3.16)

where  and  are the viscous moduli. This is the material equation of the Newtonian fluids (Newton's body). Introducing the deformation velocity spherical tensor



and the deformation velocity deviator tensor



equation (2.3.16) can be divided into two tensor equations

, (2.3.17)

where  is the so-called bulk viscosity.

In reality, to describe the frictional fluids the material equations of the Pascal’s and Newton's body should be combined, i.e. the total stress tensor is

.

By forming the divergence of the tensor and using (2.3.16) one obtains

,

with which the motion equation (2.2.5) can be written as

, (2.3.18)

or in vectorial form

.

This is the motion equation of frictional fluids, i.e. the Navier-Stokes equation.

### 2.3.4. Rheological material models and their motion equations

The material equations of Hooke’s body (2.3.4) and Newton’s body (2.3.16) in elastic aspect describes two important limit cases of isotropic material continua: the limit case of stresses depends on only the deformations (2.3.4), as well as depends on (linearly) only the deformation velocity (2.3.16) respectively. In reality, the stress tensor of the medium (in more or less scale) depends on the deformations and the deformation velocities

,

or otherwise the material equation can be written in the general form of

. (2.3.22)

In many cases, this material equation can be produced according to the equations (2.3.4) and (2.3.16) or (2.3.4) and (2.3.20), in other words the material model describing the elastic properties of the medium can be built from Hooke’s and Newton's body or Navier-Stokes’s body. In this case we are talking about a complex material model. Often, the stress change velocity tensor  plays a role in the material equation, i.e. the material equation is

. (2.3.23)

The function  is usually linear appearing in (2.3.22), (2.3.23), the so-called rheological equations. Then the stress tensor is the linear expression of tensors and  to which one will see some examples in the followings.

**The material and motion equations of the Kelvin-Voigt’s body**

The Kelvin-Voigt model shows the simpliest combination of the Hooke’s and Newton’s body. Figure 2.1. illustrates the model.



Figure 2.1.: Model of the Kelvin-Voigt’s body

The spring models the Hooke's body, while the perforated piston moving in the viscous fluid-filled cylinder models the Newton's body. In case of one-dimensional motions it is obvious that the displacements on the two body parts are equal, while the sum of the forces arose in the two branches of the model are equal to the forces affected to the model. This simple criterion can be generalized as follows

 (2.3.24)

. (2.3.25)

By using the material equations (2.3.4) and (2.3.17), (2.3.24), (2.3.25) provides the following result

 (2.3.26)

which is the material equation of the Kelvin-Voigt’s body. Based on (2.3.26) to the stress deviator tensor one obtains

 (2.3.27)

or by introducing the so-called retardation time



one gets the equation

. (2.3.28)

One can see that in case of slow processes this material equation pass through the material equation of the Hokke’s body. If  is the characteristic time of the process then  gives the order of magnitude of the derivative. In case of slow processes



then indeed . In case of fast processes

,

then the equation (2.3.28) can be approximated by



which is the deviator equation of the Newton’s body due to .

Since the deformation of the Kelvin-Voigt’s body reaches the value belongs to the Hooke’s body only delayed (retarded), the rheological parameter  is called retardation time. The rock mechanical process described above and illustrated in Figure 2.2. is called creep.



Figure 2.2.: The phenomenon of creep and the geometric meaning of parameter 

Returning to the general equation (3.2.34), by partial integration one obtains from it an initial condition independent formula

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Herewith – contrasting the Kelvin-Voigt’s body with the Hooke's body - an example can be seen to that the deformations in the body in given time  depend on not the value of the stresses in the same time but the stesses took in the previous interval .

One obtains the motion equation of the Kelvin-Voigt’s body by substituting the material equation (2.3.26) to (2.2.6).

, (2.3.35)

or in vectorial form

. (2.3.36)

**The material equation of the Maxwell body**

As it was presented, the Kelvin-Voigt body built up from the Hooke and Newton bodies behaves as linearly elastic body in case of static border-line case, and as viscous fluid in case of fast processes. Another material model can be built up from the Hooke and Newton bodies as well, which acts like fluid in slow processes and as elastic solid continuum in fast ones. This is the model of the Maxwell body illustrated sematically in Figure 2.3.



Figure 2.3.: The model of the Maxwell body

Thinking about the one dimensional motions based on the figure it can be seen that the same forces arise in the two elements of the model and the sum of the displacements of the two elements is equal to the total displacement. Generalized this, the eqations



 (2.3.37)

can be used as basic equations in the deduction of the model’s material equation. Based on the material equations of the Hooke and Newton bodies the deviator equation can be written as

, (2.3.42)

where  is the relaxation time.

A typical property of the Maxwell body can be shown if the solution relating to time stationary deformations of equation (2.3.42) is derived. There the



equation gives the result



(here the upper case  does not indicate the sphere tensor, but the value taken at !). The exponential loss of stresses is shown in Figure 2.4.



Figure 2.4.: The phenomenon of stress relaxation, the geometrical meaning of the  parameter

This phenomenon common in rocks is the release or relaxation of stresses. The  relaxation time is the time during the stresses decrease to the *e*-th part of the initial  value. The Maxwell model is basicly a fluid model, no stresses arise in it against static deformations. So it can be used only for explanation of dynamic features during describing rocks.

**The material equation of the Poynting-Thomson body**

The previously introduced Hooke, Kelvin-Voigt and Maxwell models each took hold of one important side of elastic-rheological features of rocks: the Hooke body the resistance against the static deformations, the Kelvin-Voigt body the creeping, the Maxwell body the stress relaxation. The Poynting-Thomson model or standard body is a rock mechanical model which combines the Hooke and Maxwell bodies as it is shown in Figure 2.5. and it can describe the three phenomena simultaneously. The base equations are

,

, (2.3.44)

where  are the deformation and stress arising in the Maxwell body.



Figure 2.5.: The model of the Poynting-Thomson body

The material equation of the standard body can be obtained after some algebra as

, (2.3.45)

or in an other form



where is the relaxation time is the retardation time.

As it can be seen in Figure 2.6. under constant deformation the stresses decrease from the initial  value to the  value. This is the relaxation phenomenon in case of the Poynting-Thomson body.



Figure 2.6.: Stress relaxation in case of the Poynting-Thomson body

If the stresses do not vary in time  the deformation change due to Fig. 2.7



Figure 2.7.: The phenomenon of creeping in the case of the Poynting-Thomson body

As it can be seen in Figure 2.7. the formula describes the increase of deformations from the  value to the  value. This phenomenon is called creeping. The Poynting-Thomson body can describe both the relaxation and creeping phenomena.

# 3. WAVE PROPAGATION IN ELASTIC AND RHEOLOGICAL MEDIA

The objective of geophysical investigations is mostly the determination of subsurface structures of the Earth - as material half space - by using surface measurements. For some measurement methods (gravity, magnetic, geoelectric) the impact measured on the surface is integrated in the sence that the quantity measured in a given point reflects theoretically the effect of the whole half space – but at least a space portion to a certain depth. It makes the interpretation much easier if the measured effect yields information from a local area of a determined curve and not from the whole half space. This gives the “simplicity” of the analysis of rocks by elastic waves and its importance as well, because the laws of “beam optic” can be used for the wave propagation in a certain approximation. In the followings the most important features of elastic waves are reviewed with respect to the major material equations discussed previously.

We deal only with low amplitude waves in our investigations. It means that the basic equations are solved with a linear approximation. There is a substantial derivate on the left side of the (2.2.5) general motion equation, where

,

where  convective derivative means namely a nonlinear term. Its neglect requires the fulfilment of a simple criteria in case of waves. If  is the periodic time of the wave,  is the wavelength,  is the amplitude, then the order of magnitude is



.

The convective derivative can be neglected beside the  local derivative if  , i.e. , i.e. . If this criteria is fulfilled, then we can speak about (compared to the wavelength) low amplitude waves. In this case we can write the linear  derivate instead of .

The assumption of homogeneous medium – especially if it has an infinite dimension – is unsubstantiated in geophysical aspect. We still use this approximation, because the most important properties of the wave space, the connection between the parameters characterizing the waves can be introduced most easily in case of wave propagation in infinite homogeneous medium. We do not have to consider boundary conditions during solving the differencial equations in infinite homogeneous space. It is a significant simplification. The so evolving waves are called body waves. (The assumption of infinite spreading is abstraction of course, which means the restriction that the surfaces - maybe existing in the medium - are very far from each other regarding to the wavelength.) In the followings the properties of body waves propagating in infinite homogeneous medium following different material equations are summerized.

## **3.1. Low amplitude waves in ideal fluid**

The motion equation of ideal fluid is given by (2.3.15). The low amplitude wave solution of the equation can be written in the following form

. (3.1)

From the point of view of wave theory the importance of  mass forces is confined to the determination of equilibrium  distributions. In equilibrium the



statics base equation is valid. For example in case of air this equation determines the density and pressure distributions in the atmosphere of the Earth. This distribution is inhomogeneous, but the inhomogeneity occurs on a very large scale compared to the wave length (for example the wave length of a 100 Hz frequency sound has the order of magnitude of *m*, which is really small compared to the 10 km order of magnitude of the characteristic changes of the atmosphere). So the medium is locally homogeneous from the point of view of wave propagation, i.e. the equation (3.1) can be solved for homogeneous space. If the wave propagates through distances characterized by inhomogeneity, the changes in accordance with the place of local features (local propagation velocity) should take into account.

As the  mass force field has no influence on the wave solution in the order of magnitude of wave length, we can apply the  substitution in (3.1), i.e.

.

To solve the equation system there is a need for two more equations, for the



continuity equation and for a material equation, for example the  barotropic equation of state. Assuming that the wave causes the small  changes of the  equilibrium features, i.e.



the equation system can be linearized. Neglected the product of the  quantities or their derivatives the following equations can be obtained

 (3.2)



,

where  From the last two equations

, (3.3)

the divergence of (3.2) can be written as

.

If we compare this equation against (3.3) the



wave equation can be derived. The following equation can be similarly deduced as well

.

The monochromatic plane wave solution of the equations – according to  –

can be written in the form of

,

. (3.4)

These functions satisfy the wave equation, but the question is: are they the solution of the motion equation? As we get the equation



after forming the rotation of eq. (3.2), it can be seen that the motion equation is fulfilled only in case of i.e. the displacement of the wave or rather the velocity of the displacement is parallel to the direction of wave propagation. The (3.4) function describes a longitudinal wave propagating with  velocity. This solution of the motion equation of ideal fluid (Pascal body) is the sound wave.

## **3.2. Low amplitude waves in isotropic linearly elastic medium**

The motion equation of the linearly elastic isotropic homogeneous medium is given by (2.3.13). As the  mass forces can be neglected during the analysis of the wave solution, the motion equation can be written in the following form

. (3.5)

The  vector space, which gives the displacement field, can always be decomposed into the sum of a source-free and a swirl-free vector space

, (3.6)

where

 (3.7)

. (3.8)

Using the identity



for the  vector space the following relationship can be written



Based on this equation the

 (3.9)

formula can be derived by using (3.5). As the order of the partial derivation is interchangeable,



ensues from (3.5), i.e. the first term on the left side of (3.9) is source-free as well. It can be seen similarly that the second term in the brackets is swirl-free. As the sum of a source-free and a swirl-free vector spaces can be zero only if the two vector spaces are zeros separately, from (3.9) the

 (3.10)

and the

 (3.11)

equations can be deduced, where

. (3.12)

Thus the motion equation gives a separated wave equations each for the source-free  and the swirl-free  vector spaces.

The monochromatic plane wave solution of eq. (3.10) can be written as

, (3.17)

where

. (3.18)

The  vector space satisfies the (3.7) auxiliary condition, therefore the



equation should fulfilled as well, whereof . The (3.17) describes a transverse wave propagating with  velocity. Since  these waves do not result in volume changes.

The monochromatic plane wave solution of (3.11) has the form of

, (3.19)

where

. (3.20)

The vector space satisfies the (3.8) auxiliary condition, thus

.

This criteria is fulfilled if the directions of displacement and the propagation are parallel. The (3.19) describes a longitudinal wave propagating with  velocity. It can be seen from eq. (3.12) that the latter one in the two types of waves propagating in the Hooke medium is faster

.

In comparison of the longitudinal and transverse waves originating from a common source the longitudinal waves arrive first to the observation point.

## **3.3. Small amplitude waves in Kelvin-Voigt medium**

The motion equation of the Kelvin-Voigt body can be written in the form of



based on (2.3.35) after neglecting the mass forces. In search of the wave solution with the  auxiliary condition (transverse wave), then the

 (3.26)

equation can be derived, which monochromatic plane wave solution has the form of

.

After substituting it into (3.26) a dispersion relation similar to (3.23) can be obtained

,

where . Solving the equation for the complex wave number



formulas similar to the equations (3.24), (3.25) can be deduced

 (3.27)

. (3.28)

The wave’s phase velocity  is frequency dependent, so the transverse waves propagating in the Kelvin-Voigt medium show dispersion and their absorption coefficients is frequency dependent as well. In a low frequency border-line case . Then with series expansion the equations (3.27), (3.28) can be rewritten as



.

Thus in the first approximation the



phase velocity is frequency dependent, i.e. there is no dispersion, the absorption coefficient depends on the square of the frequency. The Kelvin-Voigt medium changes to Hooke body in low frequencies in point of view of wave propagation velocity, but it preserves the properties of the Newton body with respect to the absorption.

At high frequency  In this case the equations (3.27), (3.28) lead to the results reviewed at the Newton body

.

The Kelvin-Voigt body gives back the Newton body in the high frequency border-line case. This can be expected from the structure of the model shown in Figure 2.1. The Kelvin-Voigt body is not suitable to describe weakly attenuating waves at high frequencies.

Longitudinal wave can be discussed by specifying the  auxiliary conditions. As in this case the



equation is satisfied, from the motion equation

.

Writing the time dependence in the form of  the following equation can be formulated

, (3.29)

where .

Based on the equation the



formulas can be introduced for the complex Lamé constants. With these the (3.29) can be rewritten as

.

In search of the wave solution in the form of monochromatic plane wave



based on (3.29) the following complex dispersive equation can be derived

.

For the  wave number results similar to (3.27) and (3.28) can be obtained

, ,

from which the frequency dependent phase velocity



and the frequency dependent penetration depth



can be deduced. These formulas have the form of





in the  low frequency border-line case, i.e. the phase velocity has the value characteristic for the Hooke body, the absorption coefficient picks the value characteristic for the Newton body. The attenuation is weak, as  is fulfilled trivially from the *a<<b* condition. The displacement function is

. (3.30)

It should be understandable that the Kelvin-Voigt model is not appropriate for the description of weakly attenuating longitudinal waves in high frequency border-line case, because in that case

.

Note that the Kelvin-Voigt body can be applied with different  parameters for the same rocks during description of rock mechanical and seismic phenomena. The characteristic time of creeping process in a rock has an order of magnitude of hours-days, a similar order of magnitude of  retardation time belongs to this. However for description of weak attenuation of seismic waves the  value is suitable. This fact suggests that the Kelvin-Voigt model has an approximate validity, the  parameter included in it is frequency dependent and it can be considered constant only in a narrow frequency range. However we can conclude that this model is suitable for description of weak attenuation of longitudinal and transverse waves. This is the reason that is widely used in seismics mostly for the description of wave propagation in rocks with high water and hydrocarbon content.

**The constant Q model**

The seismic experiences show that for most rocks the phase velocity is constant regardless to the frequency, but the absorption coefficient increases in direct proportion with the frequency

 (3.31)

, (3.32)

where  is the frequency independent (constant) quality factor of rocks. This rock model is called the constant Q model.

In case of Hooke body (3.31) is satisfied, but there is no absorption. For the Kelvin-Voigt body (3.31) is satisfied at low frequency border-line case, the absorption coefficient can be determined by

,

i.e. compared to (3.32) , i.e. the quality factor is not constant. Similar results are obtained (at low frequency) for the Poynting-Thomson body. The Maxwell model can describe seismic waves in high frequency border-line case, but then

,

i.e. based on (3.32) , the quality factor is proportional to the frequency. Similar results are obtained in high frequency border-line case by the Poynting-Thomson body.

It is understandable that the body following the constant Q model can be characterized by the complex Lamé constants

 (3.33)

if its material equation is assumed in the form similar to the Hooke body’s formula ( and are frequency independent)

.

The here presented dissipative rock physical parameters can be defined by the formulas

,

where is the loss angle (the angle between the stress and the deformation in case of “pure” shear(for example at transverse waves)). has similar interpretation in wave theory applications for the longitudinal waves.

Based on the stress tensor above the motion equation can be written as

 (3.34)

For transverse waves  and so from the motion equation the



equation can be derived. For the monochromatic waves written in the form of



using equation (3.33) the following dispersion equation can be deduced

,

where . For the  complex wave number the

 (3.35)

 (3.36)

equations can be written. It is easily understandable that we can speak about weak attenuation (*a<<b*) only if . Then with a simple series expansion from (3.35), (3.36) the following expressions can be derived



,

from which it can be seen after comparing to 3.32 that for transverse waves that the quality factor is really independent from the frequency

,

similar to the phase velocity .

Stipulating the  auxiliary condition for longitudinal waves the



equation can be deduced based on (3.34). In search for its monochromatic plane wave solution in the form of



the following dispersion equation can be obtained

,

where .

This equation changes into (3.35) with the substitution of , therefore its solution can be written directly according to (3.35) and (3.36)

 (3.37)

. (3.38)

Weak attenuation happens only if  (i.e. ), therefore based on (3.37), (3.38)

, .

Based on the comparison to (3.32) the quality factor for longitudinal waves can be obtained in the form

.

The constant Q model gained widespread application in the field of seismics and acoustics.

**II. Electromagnetics**

**1. The governing equations of electromagnetic phenomena**

For the understanding of natural phenomena in most cases it is unnecessary to take into consideration the atomic and molecular structure of the material, because the time interval and distance of the phenomena or process is much larger than the characteristic time interval and distance of atomic phenomena. In this case the phenomenological description can be used. Phenomenological theories provide a simplified approximate description of the material world, their advantage is that they provide a simple and exact mathematical description of the most important processes taking place on a macroscopic scale. An inevitable result of the phenomenological description is that we have to introduce the material properties (elastic constants, viscosity, thermal conductivity, dielectric constant, magnetic susceptibility, etc.). These material properties substitute those properties (atomic structure, crystal structure), which cannot be examined on macroscopic scale. The theory of electromagnetic phenomena developed based on the above is called phenomenological electrodynamics and since it is sufficient for geophysical applications we discuss only phenomenological electrodynamics in this book.

There are four basic equations, called Maxwell equations, which govern electromagnetic phenomena. The so called local forms of these equations are:

rot (1.1)

rot (1.2)

div (1.3)

div (1.4)

Here rot (curl) is the so called vortex density, is vector of the magnetic field strength, is the time derivative of the electric displacement vector , is the electric field strength, is the time derivative of the magnetic induction vector , div is the so called source density, is the charge density and is the current density vector, which consist of the convection current density and the conduction current density, however in geophysical applications convection current has no significance, therefore in the followings will denote the conduction current density.

**1.1 Boundary conditions**

The Maxwell equations are coupled linear differential equations applying locally at each point in space-time (x, t) and for solving them boundary conditions are needed. Using Stokes theorem based on Eq. 1.1 it can be seen that the tangential component of is continuous across the interface:

 (1.5)

Similarly, from (1.2) we can derive that:

 (1.6)

With the help of the Gauss-Ostrogradsky theorem, from Eq. 1.3 we can derive for vector’s normal component the following boundary condition across the interface:

where is the surface charge density. Based on Eq. 1.4 we can set the boundary condition that the component of the magnetic induction vector is continuous across the interface:

.

**1.2 Material equations**

 in the Maxwell equations and the are not independent. In vacuum we can define the

 and (1.7)

equations, where is the dielectric constant or permittivity of vacuum and is the magnetic permeability of vacuum.

If is the electric- is the magnetic polarization vector (which mean the electric and magnetic dipole moment per unit volume respectively), then we can use the following definitions for and :

 and . (1.8)

The relationship of and vectors and the relationship of and vectors are investigated by taking into consideration the atomic structure of the material by the electron theory, statistical mechanics and quantum physics. In phenomenological electrodynamics we simplify this relationship and we do not deal with the atomic nature of things. It is the easiest to assume a linear relationship:

 and , (1.9)

where and are the electric and magnetic susceptibility of the medium respectively, or based on Eq. (1.8):

 (1.10)

and

 , (1.11)

where and are the permittivity and the permeability of the medium respectively. The dielectric properties of the medium characterized by susceptibilities and permeabilities are taken as known material constants in phenomenological electrodynamics or in case of inhomogeneous medium, as a function of the space coordinates. However, these “material constants” are temperature dependent and often even depend on the frequency of the electromagnetic field. (This phenomenon can be explained by taking into consideration the inner structure of the material). The linearity of Eq. 1.10 and 1.11 are usually adequately fulfilled, which means that and are independent from the field strengths. For ferroelectric and ferromagnetic materials, the relationship of and cannot be given by a linear function, and it is not even a single valued function (hysteresis).

For materials which show anisotropy on macroscopic scale, Eq. (1.10-1.11) are not valid, therefore the following more general material equations are used instead:

 and ,

where is the dielectric, is the magnetic permeability tensor. In this case usually the , vectors are not parallel.

In a conductive medium the current density is usually not pre defined, but is defined by the strength of the electric field. For homogenous and isotropic bodies, this relationship is given by the differential Ohm's Law:

 , (1.12)

where is the electrical conductivity. This material property also depends on temperature, frequency and on other parameters connected to the inner structure of the material. However, phenomenological electrodynamics does not deal with these effects, electrical conductivity is considered a known constant.

**1.3 The special phenomena of electrodynamics**

With the Maxwell equations all the electromagnetic phenomena can be described in a theoretical way. There are however some phenomena, which do not require the usage of the (1.1) – (1.4) equations in their general form. Because of the high number of these phenomena, it is common to divide electrodynamics into different chapters.

We deal with statics if the physical quantities are constant with time, the charges are in permanent magnetic state and there is no flowing current. In the Maxwell equations then and

The basic equation of electrostatics:

rot

div

The basic equations of magnetostatics:

rot

div

We talk about stationary currents when the flowing currents in the conducting medium are independent from time. It is easy to see that in this case there is no volume charge density in the Maxwell equations. The continuity equation of charge:

 ,

and using the differential Ohm’s law , based on Eq(1.3) we get

 .

If we assume that inside the conducting media at t= there is volume charge density, then by solving the above equation we get

 ,

where is the relaxation time of the medium. This equation indicates that the volume charge density decreases exponentially. (During the relaxation time it decreases by ). This can be explained easily: the charges inside the conductor can move and because of their repelling effect they get to the surface of the conductor. Relaxation time describes this process. Rocks that are important from a geophysical point of view, limestone has the lowest conductivity ( which results in . For other rocks is even smaller.

So even if there are volume charges in the conductor, they get to the conductor’s surface in a very short time and their effect only last for the relaxation time. After a longer period, the volume charge density is zero. Therefore, in case of stationary processes the basic equations can be written as:

rot ,

div ,

,

We talk about Quasi-Stationaryprocesses when the displacement current density is negligible beside the current density flowing in the medium. The condition for this can be easily derived for phenomena that are time dependent by . Let’s assume that in the medium the electric field strength is . Then according to the differential Ohm’s law:

 ,

and the displacement current density

 .

The magnitude comparison lead to the relation.

This can also be written as , where is the period, is the relaxation time as described earlier. It is obvious that volume charges cannot be in the medium in this case either. The cutoff frequency in case of rocks is the smallest for limestone . However, this is a very high frequency, the condition is practically always fulfilled in geophysical application. The basic equations of the quasi-stationary currents’ field:

rot ,

div ,

,

For fields that are changing rapidly with time the displacement current density in Eq.(1.1) needs to be taken into consideration as well. In this case for describing electromagnetic phenomena we use the system of Maxwell equations (1.1) – (1.4).

**2. Electromagnetic potentials**

The Maxwell equations are coupled partial differential equations. Their general solution cannot be written directly. Therefore, any method that simplifies the solution of the field equations is very useful. One of this method was the introduction of electromagnetic potentials.

Equation (1.4) can be trivially satisfied if we take the magnetic induction vector in the following form:

 , (2.1)

where vector field for the time being is the unknown vector potential. With this (1.2) can be written as

 .

This equation can be trivially satisfied, if

 ,

where is the presently unknown scalar potential which is an arbitrary, continuous function that is differentiable. With the vector- and scalar potential the electric field can be given as:

 . (2.2)

It can be seen that with the four introduced scalar function (), (2.1), (2.2) and i.e. six scalar fields can be defined.

For the unknown potentials we can derive relationships based on (1.1) and (1.3) and on the material equations (1.10) and (1.11). Assuming a homogenous medium ( based on (1.1) we get the equation

 , (2.3)

where we used the

identity. Similarly based on (1.3) we get the equation

 . (2.4)

(2.3) and (2.4) are second order coupled linear partial differential equations of electromagnetic potentials. The solution of these mathematically is just as complex as the solution of the Maxwell equations. A significant simplification is possible if we examine the clear determination of potentials.

**2.1 Gauge transformation, Lorentz gauge condition**

It can be seen easily that the electromagnetic potentials with the (2.1), (2.2) equations are not clearly determined. Let’s form the

 (2.5)

new potential, where is an arbitrary function. Based on (2.1) we can see that the vector space calculated with this is as follows:

and it equals , since So it clearly determines the vector potential (2.1) only to the extent of the gradient of an arbitrary function. However, the vector potential modified or transformed according to (2.5) produces the same B vector space.

Let’s modify the scalar potential according to:

 , (2.6)

then we get the

field strength, so according to (2.2)

The modification of the electromagnetic potentials according to (2.5), (2.6) is called gauge transformation. This transformation leaves the and fields unchanged, so the system of Maxwell equations are not affected by this transformation. In other words, the Maxwell equations are invariant under the gauge transformation.

The determination of potentials by (2.5), (2.6) is unclear, therefore additional restrictions need to be defined. These restrictions are given automatically by (2.3) and (2.4), because if we specify the equation

 , (2.7)

then Eq. (2.3), (2.4) are simplified. Eq. (2.7) imposed on the potentials is called the Lorentz gauge condition.

**2.2 Potential equations, retarded potential**

If the (2.7) Lorentz gauge condition is fulfilled, then Eq. (2.3) and (2.4) take the following forms:

 . (2.8)

 . (2.9)

These equations are inhomogeneous wave equations, or the so called d’Alembert differential equations. Having the Lorentz gauge condition, the potential equations become uncoupled thus the components of the vector potential and the scalar potential now can be determined independently from each other.

The solution of Eq. (2.9) is given by the sum of the general solution of the homogeneous equation and one particular solution of the non-homogenous equation. The solution of the homogenous equation (wave equation) will be discussed later. The particular solution can be written as follows.:

 , (2.10)

where , and the integration needs to be extended to that V’ part of the field where the charges are located.

According to the (2.10) expression the value of potential at point P of the field denoted by the vector at t time can be determined by the summation (integration) of the unit potentials deriving from the charges located in the dV’ unit volumes that are situated around point P’.



Figure 1.

The integration needs to be extended to that V’ part of the field where the charges are located. At time t, the unit potential in point P is determined by the value of charge density of point P’ not at t but at time . Since is the distance between point P and P’, v is the velocity of propagation of the electromagnetic effect, the difference between the two times equals the time, in which the electromagnetic effect gets to point P from P’.

Because of the electromagnetic effect’s finite velocity of propagation, the effect of the charge density change in point P’ occurs later () in point P (this delay is called retardation). The scalar potential (2.10) takes into consideration this retardation, and that is why the solution of Eq. (2.9) is called the retarded potential.

The solution of Eq. (2.8) can be written similarly

 . (2.11)

Based on (2.10) and (2.11) the sources are known, the retarded potentials and through (2.1), (2.2) the electromagnetic fields can be determined.

**3. The wave equation and its solutions**

We have already examined the particular solutions of Eq. (2.1) and (2.11), the retarded potentials. For the complete solution of the equations, the general solutions of the homogenous equations are also necessary. These equations jointly can be written as

 , (3.1)

where denotes one of followings: and . Eq. (3.1) is called the wave equation. It is easy to see that in case of homogeneous isotropic insulators, a wave equation can be directly deduced for the field strengths (🡪. The wave equations are present in other phenomena (acoustic, seismic) as well, then denotes e.g., pressure, density or displacement. In homogenous media the c quantity in Eq. (3.1) is constant. For the clear solution of the equation both initial- and boundary conditions need to be set. Finding a solution that satisfies these condition is usually a quite challenging mathematical task. To simplify the solution let’s assume that the source in the homogenous space is extremely far away, then we get the plane wave solution.

**3.1 The plane wave solution of the wave equation**

We get a particular solution of the wave equation (3.1) with the transformation of the independent variables

 (3.2)

 , (3.3)

where , are real constants. Based on (3.2) and (3.3) it can be seen that

 ,

and thus

 , j=1,2,3. (3.4)

The sign indicates that the differentiation inside the parenthesis needs to be done twice.

Taking into consideration Eq. (3.4), the wave equation leads to

 .

If the equation

 (3.5)

is fulfilled, then

or otherwise

 . (3.6)

This equation can be trivially satisfied, if we take the function in the following form

 , (3.7)

where are arbitrary functions that can be differentiated at least twice. The solution of the wave equation (3.7) is called the d’Alambert-solution.

With the notation , Eq. (3.5) can be written as

 .

Based on this the unit vector can be introduced with the following components

By using this, Eq. (3.2) and (3.3) can be written in the form of

or otherwise

 (3.8)

 , (3.9)

where . So the d’Alambert-type solution of the wave equation is

 (3.10)

where are arbitrary functions, and between k the equation

 (3.11)

is fulfilled.

The (3.8) particular solution can be generalized easily. If

are the two independent solutions of the wave equation, then because of the linearity of the equation

also satisfies the wave equation. Based on this the equation

 (3.12)

is also a solution. If the parameter is continuously distributed in the [ interval, then as the superposition of the particular solutions the

functions produced with integration are also solution of Eq. (3.1). Since the wave equation contains time- and space coordinates derivatives, in the argument of the functions the parameter can appear separately

 , (3.13)

where we applied the notations (3.8) and (3.9).

The (3.12) and (3.13) generalization can also be done with the parameters:

 .

(3.14)

The solution in (3.10) at fixed time gives the function’s (which characterizes the physical state) constant values on the

surface, which is the equation of a plane ( is a position vector pointing to a fixed point of the space). Therefore, the function given in (3.10) in a general sense describes a plane wave. If in the description of a specific phenomenon from the space coordinates only plays a role, then (3.10) can be written as

 . (3.15)

The function denotes the wave propagating in the positive direction on the axis and the function denotes the wave propagating in the negative direction of axis. Since the coordinate system can be chosen arbitrary, play the same role. Therefore in the followings it is sufficient just to deal with the part of the solution.

Until now the denoted an arbitrary chosen function. However, when dealing with waves, we usually come across functions that are periodic in time and space. For example:

 , (3.16)

where is the constant amplitude, is the initial phase. The (3.16) function is the monochromatic plane wave solution of the wave equation. At a fixed time in the positions with the same phases in the space they are located on the plane denoted by the equation:

 .

, is the normal unit vector of the plane wavefront.

If the period is T, then according to (3.16) the points of the wavefront related to , at time are in the same physical state, the phase however

 ,

from where

 . (3.17)

The constant in the (3.16) monochromatic plane wave solution is connected to time periodicity, it is called angular frequency.

In fixed according to (3.16) infinitely many planes can be found where the physical state is the same (the value of is the same). The distance of two adjacent wavefronts is the wavelength which is characterized by the spatial periodicity. On the wavefront then the phase difference is . If the adjacent plane’s points are given by

 ,

then the phase

 ,

from where

k= . (3.18)

The constant k in (3.16) therefore is in connection with the spatial periodicity, and is called wavenumber and is the wavenumber vector.

The wavefront is propagating in space. Directing the normal vector of the surface parallel with the propagation, with time the points of the

surface characterized by phase will move to the points of the wavefront

 .

(During the propagation of the wavefront remains unchanged). We interpret the displacement vector so that it is parallel with the vector. Then and thus

 ,

from where the velocity of propagation, the phase velocity is

 .

Comparing this result with Eq. (3.11) we can see that , which means that the c constant in the wave equation gives the phase velocity of plane waves propagating in infinite space. The function given in (3.16) is the real part of the complex function

 , (3.19)

which is mathematically also a solution of the wave equation according to (3.10). To efficiently utilize the tools of the complex function theory, it is advisable to use the complex function (3.19) instead of (3.16). Obviously only the real part of complex expressions describing physical quantities have physical meanings.

Introducing

complex amplitude, (3.19) can be written as

 . (3.20)

This is the monochromatic plane wave solution of the wave equation in complex form. Based on (3.12) the generalized plane wave solution can be written in the form of

 .

Starting from the d’Alambert type solution (3.13) of the wave equation, its general form can be written through its particular solution (3.20) as:

 . (3.21)

This equation shows the importance of the monochromatic plane wave solution, because it indicates that any wave phenomenon that varies arbitrarily over time (e.g. pulse) can be constructed as the superposition of monochromatic plane waves. Eq. (3.21) is also the Fourier-integral solution of the wave equation. This can be further generalized according to Eq. (3.14):

 , (3.22)

where .

Equation (3.11) shows the simple relationship between the angular frequency and the wavenumber k:

 ,

where c=constant. Generally the equations which give the relation are called dispersion relations. The dispersion equations can also be complex and the and k quantities in them are not necessarily real either. Later we will see this for example in the description of attenuation of waves in space and time.

If the quantity in the wave equations is complex, then if Eq. (3.5) is fulfilled then as the solution of the wave equation we get function (3.7) again. According to Eq. (3.11) with the complex k wavenumber, (3.10), (3.16), (3.19) and (3.21) are still solution of Eq. (3.1).
If , then based on (3.19)

or otherwise

ψ .

If , this equation describes a wave with phase velocity that exponentially decreases its amplitude. The amplitude decreases from the initial value to , while the wave propagates distance in the medium.

The d distance is the penetration depth. So the imaginary part of the complex wavenumber characterizes the attenuation of the wave, the quantity is called the absorption coefficient.

**3.2 The spherical wave solution of the wave equation**

It often occurs that the examined wave phenomenon shows spherical symmetry (e.g. field of an isotopically transmitting point source). Then in spherical coordinate system and

 .

Introducing the notation, the wave equation (3.1) will have the form:

 . (3.23)

Taking into consideration that the

one-dimensional wave equation’s solution according to (3.15) can be written as:

 *.*

 Therefore with the substitution (provided that (3.11) is fulfilled) the solution of (3.23) can be directly written as:

 .

Thus the spherical wave solution of the wave equation is

 . (3.24)

As we saw in (3.15) the plane wave solution, the two particular solutions (the wave propagating along the axis to the right or left direction) physically had the same weight. However the difference between the two particular solutions in the spherical wave solution (3.24) is physically very important. The particular solution describes a spherical wave propagating (divergent) out of the origin (source). This solution has a direct physical meaning. The particular solution describes a spherical wave propagating into (convergent) the origin from the infinity. This should be transmitted from a spherical surface which has an infinite radius (as a source). Obviously this solution is physically not possible. However mathematically it is a solution of the wave equation and for the solution of some problems we use spherical wave series expansion.

The monochromatic spherical wave solution of the wave equation can be written as:

 (3.25)

or in a complex form:

 , (3.26)

where

complex amplitude. In (3.25) and (3.26) the constants and k are the same as for the plane wave solution (3.17) and (3.18). The points of the spherical surface with the same phase

 ,

with time later will be on the wavefront

 ,

therefore

 .

The propagation velocity of the wavefront – the phase velocity – thus

 .

So based on (3.11) again.

The superposition of the (3.26) monochromatic spherical wave solutions also satisfies the wave equation:

. (3.27)

In this case we have a composite spherical wave solution. If the parameter is continuously distributed in the [ interval, then (3.27) can be written in the more general form:

 . (3.28)

In this solution the function is arbitrary. However it is obvious that physically is in a relationship with the source of the wave.

**4. Electromagnetic waves**

Electromagnetic waves transmitted from artificial or natural sources are useful tools of the exploration of geological structures. For geophysical applications, primarily the wave propagation in the conducting medium needs to be studied. In this chapter we will discuss the electromagnetic waves propagating in infinite, homogenous medium and along infinite conducting half-space when the source is extremely far away. Then we will derive the fields of electric and magnetic dipoles transmitting in homogenous conductors and insulators. In order to show the similarities and differences between the properties of electromagnetic waves propagating in conductors and insulators, we will shortly summarize the most important facts about insulators.

**4.1 Electromagnetic waves in homogenous, isotropic infinite insulator**

Through Maxwell’s discovery (He added displacement current to the electric current term in Ampère's Circuital Law) the structure of the system of basic equations describing the electromagnetic field transformed in such a way that wave equations can be derived from them. Maxwell realizing this predicted the theoretical existence of electromagnetic waves.

In non-conducting medium (insulator) with the assumption , taking the curl of Eq (1.1) we get the

equation, which based on (1.2) and the material equations leads to the homogenous wave equation:

 . (4.1)

Similarly, for the electric field strength:

 . (4.2)

The monochromatic plane wave solution of the equations based on (3.19) can be written directly as:

 . (4.3)

 . (4.4)

These functions do not satisfy the Maxwell equations directly. According to (1.3) in case of

 . (4.5)

In the functions (4.3) and (4.4) the space coordinates have the form of therefore

 that is,

Thus based on (4.5) we get the

 or

equation. The (4.3) function, which satisfies the wave equation, satisfies the (1.3) Maxwell equation only if . Similarly, based on (1.4) and from (4.4) we get the condition or .

Based on the (1.2) Maxwell equation and (4.3), (4.4)

 .

Since

 (4.6)

therefore

or because of

from where and

 . (4.7)

Between the magnetic and electric filed strength vectors there are no phase difference, their amplitude vectors are perpendicular not just to each other but to the unit vector pointing to the direction of propagation as well. Based on (4.7)

 . (4.8)

The energy density of the electric field

and for the magnetic field the energy density is

 .

Based on (4.8) it can be seen that

 . (4.9)

Thus the energy density of the field is .

The energy current density vector of the electromagnetic field is

based on (4.7)

 (4.10)

where

**4.2 Electromagnetic waves in homogenous, isotropic infinite conductor**

The most important properties of the propagation of electromagnetic waves in conductors can be derived in the plane wave approximation. The infinite plane wavefront is an ideal borderline case, which is fulfilled in practice satisfactorily, if the specific size of the finite wavefront and the radius of curvature of the surface are very large compared to the wavelength. The advantage of the plane wave approximation is that the field parameters can be defined with simple mathematical tools.

In homogenous conductor the differential Ohm’s law has the form of , where is the scalar conductivity which is independent of location.

With this the curl of the (1.1) Maxwell equation can be written as

 . (4.11)

Using the material equations (are constants) and with the assumption the

Maxwell equations, based in (1.2) and (4.11) can be brought to the

 (4.12)

form. Taking the curl of Eq. (1.2) with a similar process, for the electric field strength we get the equation

 . (4.13)

So in homogenous, isotropic conductors the field strength vectors satisfy the (4.12), (4.13) telegraph equations. Looking for the plane wave solution of these equations, the time dependency of the field parameters can be assumed in the form of .

Then the

relationships are met, which (4.12), (4.13) formally can be transformed to wave equations:

 (4.14)

 , (4.15)

where . Here we would like to mention that the medium, describing phenomena showing time dependency, can be characterized by the complex dielectric constant

introduced in equations (4.14), (4.15). Thus we also introduce the complex phase velocity. The solutions of Eq. (4.14), (4.15) can be found with the method presented in chapter 3, however the k wavenumber introduced with the

 (4.16)

equation, now is complex

 . (4.17)

The monochromatic plane wave solutions written based on (3.19) will look as:

 (4.18)

 (4.19)

(Since with the used complex method

 (4.20)

the (4.18), (4.19) functions substituted into the equations (4.14), (4.15), we indeed get an equation which correspond with Eq. (4.16)

 . (4.21)

From the equations (4.17) and (4.21) for the imaginary and real parts of the complex wave number we get the

equations, from where

 .

From the equation’s roots

only is real, therefore the solutions of (4.21) are

 (4.22)

 . (4.23)

In these equations the – and + signs are chosen depending on the coordinate system. Giving the electric field strength in the following form:

we can see that if a > 0

the function

 (4.24)

in case of an attenuating and in case of describes a wave that exponentially increases in amplitude. Thus we get the physically acceptable solution in (4.22), (4.23) in case of ()>0 by choosing the (+) and in case of choosing the (-) sign.

In one dimensional case (4.24) can be written as

 .

In case of a << b, this solution describes a wave propagating in the direction of the axis with phase velocity, which amplitude attenuates by the function. If d denotes the distance, along which the amplitude measured at the position attenuates by then ad=1, so

. (4.25)

The d distance is characteristic of the attenuation of the wave. Approximately it gives the wave’s depth of penetration in conducting medium. It is also called skin depth. As it can be seen in (4.25) it mainly depend on frequency and conductivity.

The phase velocity beside the material properties also depends on the frequency as it can be seen in the expression:

. (4.26)

So wave propagation in conducting medium is dispersive.

The (4.18), (4.19) functions satisfy the (4.12), (4.13) telegraph equations derived from the Maxwell equations, if the complex k=b-ia wavenumber is the solution of Eq. (4.21). However, the field strengths also have to satisfy the Maxwell equations. It is obvious that it means further restrictive conditions.

Eq. (4.18) satisfies the equation if the equation

is fulfilled, which leads to the condition even though k is complex. Similarly, the equations leads to . So the field strength vectors even for electromagnetic waves propagating in conducting medium are perpendicular to the direction of wave propagation.

From the (1.2) Maxwell equation, using (4.6) we get the condition

 . (4.27)

Using (4.22) and (4.23), the complex wave number can be written with Euler's formula as:

where

 . (4.28)

With this based on (4.27) we get to the result:

 ,

from where

 (4.29)

and

 . (4.30)

The electric and magnetic field strength vectors of the electromagnetic waves propagating in conducing medium are perpendicular to each other and to the direction of propagation as well, meaning that the waves are transverse. Between the amplitudes the relationship is the following:

 . (4.31)

According to (4.30) between the field strengths there is phase difference, which based on (4.28) can take the values , while the conductivity varies (and thus the as well) on the (0,) interval. So that magnetic field strength in homogenous conductors always delays compared to the electric field strength. The phase difference is 45° at most.

In case of

 (4.32)

 . (4.33)

Now let’s calculate the energy density of the electromagnetic wave propagating in a conductor

where

denote the real parts of the (4.18), (4.19) complex electric field strengths, is the notation of complex conjugate. The (4.32) energy density is a fast varying function of time and space, with measurements we usually determine its time average. But since in the expression

 depends on time , depends on time as , during time averaging only the time independent member remains:

 .

So the average of the energy density by using

or by using (4.31) is:

 ,

where

 is the average of the electric energy density. Introducing the average magnetic energy density we can see that

 , (4. 34)

which means that in electromagnetic waves propagating in conducting medium, the magnetic energy density is always greater than the electric energy density.

The results derived for electromagnetic waves propagating in conducting medium can be further studied in two boundary cases. The first boundary case leads to the known equations of insulators. In this case in (4.14), (4.15) , so there is no dispersion. According to (4.22) , and according to (4.23) a=0 which means that the wave attenuates. From Eq. (4.28) we get , which means that there is no phase difference between the filed strengths vectors, and Eq. (4.29) in case of returns Eq. (4.7).

The other boundary case is the high conductivity boundary case . As we have seen in the subchapter 1.4, this is the Quasi-Stationary or low frequency approximation: is the relaxation time. We can also mention that in this case the displacement current density is much smaller than the current density. The condition, for limestone which has the lowest electric conductivity is approximately fulfilled up to (for other rocks even higher). Since the depth of penetration at in limestone is (for other rocks even smaller), the frequency used in measurements needs to be much smaller than 1MHz. Thus the condition is always fulfilled in geophysical application. Then the electric and magnetic field strengths based on (4.32) and (4.33) can be written as:

 ,

where

 . (4.35)

(So the attenuation is not weak: b=a), and based on (4.28) . In this bordierline case the penetration depth of the electromagnetic wave is

 .

The relationship between the amplitudes of the field strengths is given based on (4.31) by the equation:

 .

Between the time averages of the magnetic and electric energy densities the following relation is fulfilled:

 .

The functions (4.32), (4.33) which we have got as the solutions of the telegraph equation, give the field strengths of the electromagnetic plane waves propagating in conductive medium. The main property of the waves, that they attenuate and are dispersive. We can experience the attenuation of electromagnetic waves in everyday life for example when the radio quiets down in tunnels and in reinforced concreate buildings. The results we have got can also be confirmed by some optical examples. Glass is a good insulator and in it electromagnetic waves (light as well) do not attenuate much, therefore the glass is transparent. Metals are good conductors, therefore they are opaque to electromagnetic waves, especially for light. There are some counterexamples which are at first glance puzzling, ebonite, Bakelite and caprolactam are good insulators, so we would expect them to be transparent, but they are not. On the other hand, even though salt crystal is a relatively good conductor, it is not opaque.

This contradiction is solved by the fact that in our derivations the “material constants” were assumed to be constants, but they are a function of frequency. This frequency dependence should not be overlooked, for example if we would like to utilize our results which we got from 50Hz current, in optical frequencies of . Thus for example the ebonite, Bakelite etc. known as insulators above behave as good conductors and salt is an insulator in optical frequency.

**4.3** **Electromagnetic waves along infinite conductive half-space**

Up until now we have dealt with wave propagation in infinite media. We assumed the wave source to be infinitely far away, thus we did not have to deal with the initial and boundary conditions. However, in geophysical application usually we have a layered medium. Then the solutions of the wave equations at the interfaces have to satisfy the boundary conditions shown in subchapter 1.1. The easiest assumption which makes it possible to study the most important properties of waves propagating in layered medium is to take an infinite conductor contacting a nonconducting half space. Let half-space be an insulator , the half-space be a conductor, with the material properties and let’s assume that the wave is arriving along from a source infinitely far away. Then the parameters of the fields depend on time and on the coordinate as , and the coordinate does not affect the description of the phenomenon , so

 . (4.36)

 . (4.37)

Using the material equations as well with this the (1.1) and (1.2) Maxwell equations can be written in the form of:

 .

 .

By expounding the equations, we get two independent system of equations:

 (4.38)

 (4.39)

 . (4.40)

and

 (4.41)

 (4.42)

 . (4.43)

From the two equation groups it is enough just to solve one of them, e.g. the (4.38)-(4.40) system of equations, because from this the field parameters () 🡪() and with the substitution of the material properties , we get the (4.41)-(4.43) equations.

Expressing the functions from the equations (4.38) and (4.39)

 (4.44)

 (4.45)

and substituting into (4.40) we get the equation

 , (4.46)

where we introduced the

 (4.47)

notation. The general solution of the (4.46) differential equation can be simply written as:

 , (4.48)

where A and B are constant of integration.

The expression returns two complex numbers, which are each other’s complex conjugate. Both roots are suitable to describe the field strength. From now on we will use the one which has the positive imaginary part. Then in the half-space in (4.48) we have to choose B=0, because the expression in case of approaches to infinity, but the field strength can only have a finite value. Similarly, in the half-space only in case of A=0 we get a regular solution. Thus we get to the

result, where . However, based on the (1.5) boundary condition

 ,

from where A=B, thus the (4.46) equation is regular in and its solution on the plane fulfilling the boundary conditions as well:

 . (4.49)

Using this and according to (4.44) and (4.46)

 (4.50)

and based on (4.45) we get the

 (4.51)

result. The k wavenumber in the equations (4.49) -(4.51) is unknown. The (1.6) boundary condition

based on (4.50) on the plane, provides the fulfillment of the equation:

 ,

from where

 . (4.52)

This equation is the dispersion relation of the electromagnetic waves propagating along infinite conductive half-space, which with the help of (4.47) can be written as:

 ,

where .

In the high conductivity borderline case and thus

 .

Then in the half-space, based on (4.49), (4.50), (4.51) we get to

 ,

where, . So in insulator we get a transversal electromagnetic wave propagating without attenuation nor dispersion with c phase velocity, which amplitude is independent from the coordinate. This solution equals the solution we have got for infinite insulator. Paradoxically, if the conducting half-space is an extremely good conductor ( then it has no effect to the insulator half-space.

In the half-space

 ,

therefore and thus for the field strengths we get the equations:

 .

 .

 .

However, based on (4.47) it is easy to see that in the high conductivity borderline case

and thus the above expression can be written in the form of:

 . (4.52)

 . (4.53)

 . (4.54)

In the conducting half-space, the field strengths are exponentially decreasing as we get farther from the interface. The depth where it decreases to its part is the skin depth:

 (4.55)

This distance gives the approximate penetration depth of electromagnetic waves into conducting medium. In the high conductivity borderline case the field strengths are perpendicular to each other and the phase difference between them is .

The (4.52)-(4.54) expressions contain the space coordinates in the form of , where .

So the wave in the conducting half-space propagates in the plane. Denoting the angle between the direction of propagation and the axis with

 .

Since , therefore . So the wave propagates almost parallel with the axis.